

# Associated graphs of le-modules

<sup>1</sup>Sadashiv Puranik,

and

<sup>2</sup>Sachin Ballal

and

<sup>2</sup>Vilas Kharat

<sup>1</sup>Tuljaram Chaturchand College of Arts, Science and Commerce, Baramati-413 102 (INDIA)

<sup>2</sup>Department of Mathematics, Savitribai Phule Pune University, Pune-411 007 (INDIA).

---

Let  $M$  be an le-module over a commutative ring with unity. In this paper, an associated graph  $G(M)$  of  $M$  with all nonzero proper submodule elements of  $M$  as vertices has been introduced and studied. Any two distinct vertices  $n$  and  $m$  are adjacent if  $n + m = e$ . Some algebraic, topological and, graph theoretic properties of le-modules have been established. Also, it is shown that the Beck's conjecture is true for coatomic le-modules.

Keywords: submodule elements, superfluous element, complete graph, connected graph

---

## 1. INTRODUCTION:

A. K. Bhuniya and M. Kumbhakar [(Bhuniya and Kumbhakar, 2018, 2019)] introduced and studied a new algebraic structure, which is called an le-modules. An le-module  $M$  over a commutative ring  $R$  is a complete lattice ordered monoid  $(M, +, \leq, e)$  with greatest element  $e$  and module like action of  $R$  on it. A. K. Bhuniya and M. Kumbhakar motivated to study abstract submodule theory from the study of abstract ideal theory, in particular multiplicative lattices and lattice modules. For more details about multiplicative lattices and lattice modules one may refer [(Narayan Phadatare and Kharat, 2019)], [(Ballal and Kharat, 2015)]. The notion of a graph of zero-divisors of a commutative ring was introduced in [(Beck, 1988)], by studying the coloring of a graph constructed by all elements of a commutative ring  $R$ . In [(A. Abbasi, 2015)] A. Abbasi, H. Roshan-Shekalgourabi, D. Hassanzadeh-Lelekaami introduced and studied associated graph on modules over commutative rings. Elham Mehdi-Nezhad and Amir M. Rahimi studied similar type of graph and proved some new results on it. Narayan Phadatare, Sachin Ballal and Vilas Kharat studied graph on multiplication lattice modules by using a non-small element. Also they have introduced semi-complement graph on lattice modules.

In this paper we have introduced and studied associated graph on le-module. We have obtained analogous results as in paper [(A. Abbasi, 2015)] and for it we got some new algebraic results for le-modules.

### 1.1 Definition:

[(Bhuniya and Kumbhakar, 2018, 2019)] An le-semigroup  $(M, +, \leq, e)$  is a commutative monoid with the zero element  $0_M$  and is a complete lattice with the greatest element  $e$ , that satisfies  $m + (\bigvee_{i \in I} m_i) = \bigvee_{i \in I} (m + m_i)$ . Let  $(M, +, \leq)$  be an le-semigroup with the zero element  $0_M$  and  $R$  be a commutative ring with unity  $1_R$ . Then  $M$  is called an le-module over  $R$  if there is a mapping  $R \times M \rightarrow M$  satisfying:

- (1)  $r(m_1 + m_2) = rm_1 + rm_2$
  - (2)  $(r_1 + r_2)m \leq r_1m + r_2m$
  - (3)  $(r_1r_2)m = r_1(r_2m)$
  - (4)  $1_Rm = m ; 0_Rm = r0_M = 0_M$
-

- (5)  $r(\vee_{i \in I} m_i) = \vee_{i \in I}(r m_i)$  holds for all  $r, r_1, r_2 \in R, m, m_1, m_2 \in M$  and  $i \in I$  ( $I$  is an indexed set).

A *graph*  $G$  is defined as the pair  $(V(G); E(G))$ , where  $V(G)$  is the set of vertices of  $G$  and  $E(G)$  is the set of edges of  $G$ . For two distinct vertices  $n$  and  $m$ ,  $n - m$  means that  $n$  and  $m$  are *adjacent*. The *degree* of a vertex  $n$  of graph  $G$  which is denoted by  $deg(n)$  is the number of edges incident on  $n$ . If  $|V(G)| \geq 2$ , a *path* from  $n$  to  $m$  is a series of adjacent vertices  $n - v_1 - v_2 - \dots - v_n - m$ . In a graph  $G$ , the *distance* between two distinct vertices  $n$  and  $m$ , denoted by  $d(n; m)$  is the length of the shortest path connecting  $n$  and  $m$ . The *diameter* of a graph  $G$  is defined as  $diam(G) = sup\{d(n; m) | n, m \in V(G)\}$ . A graph  $G$  is called *connected*, if for any vertices  $n$  and  $m$  of  $G$  there is a path between  $n$  and  $m$ . The *girth* of  $G$ , is the length of the shortest cycle in  $G$  and it is denoted by  $g(G)$ . A graph is called *complete* if each pair of vertices is adjacent. A complete graph with  $n$ -vertices is denoted by  $K_n$ . An  $r$ -*partite* graph is one whose vertex set can be partitioned into  $r$  subsets such that no edge has both ends in any one subset. A *complete  $r$ -partite* graph is one each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e, 2-partite) graph with part sizes  $m$  and  $n$  is denoted by  $K_{m,n}$ . A *clique* of a graph is its maximal complete subgraph and the number of vertices in the largest clique of a graph  $G$ , denoted by  $\omega(G)$ , is called the *clique number* of  $G$ . A graph whose vertices-set is empty is a *null graph* and a graph whose edge-set is empty is an *empty graph*. Color- ing of a graph  $G$  is an assignment of colors (elements of some set) to the vertices of  $G$ , one color to each vertex, so that adjacent vertices are assigned distinct colors. If  $n$  colors are used, then the coloring is referred to as an  $n$ -*coloring*. If there exists an  $n$ -coloring of a graph  $G$ , then  $G$  is called  $n$ -*colorable*. The minimum  $n$  for which a graph  $G$  is  $n$ -colorable is called the *chromatic number* of  $G$ , and is denoted by  $\chi(G)$ . The *core* of a graph  $G$  is the union of cycles in  $G$ . A vertex  $x$  of a graph  $G$  is called an *end vertex* if  $deg(x) = 1$ . For further study of graph theory see [(Bondy and Murty, 2008)].

1.2 Definition:

An element  $n$  of an le-module  $M$  is said to be a *submodule element* if  $n + n, rn \leq n$  for all  $r \in R$ . We denote the set of all submodule elements of  $M$  by  $Sub(M)$ .

Submodule elements are the ones on which the theory of an le-module is being studied. Observe that if  $n, m \in Sub(M)$  then  $n + m \in Sub(M)$  and  $rn \in Sub(M)$ , and note that  $n + n = n$  for all  $n \in Sub(M)$ .

1.3 Proposition:

Let  $M$  be an le-module. Then

- (1)  $n \geq 0$ , for  $n \in Sub(M)$ .
- (2)  $n + m \geq n \vee m$  for  $n, m \in Sub(M)$ .
- (3) If  $n + m \neq n \vee m$  implies  $n \not\leq m$  and  $m \not\leq n$  for  $n, m \in Sub(M)$ .
- (4) If  $n \leq m$  then  $k + n \leq k + m$  for all  $n, k, m \in M$ .
- (5) If  $n, m \in Sub(M)$  then  $n \wedge m \in Sub(M)$ .

PROOF. (1) We have  $rn \leq n$  for all  $r \in R$ . Hence  $0n = 0 \leq n$  for  $n \in Sub(M)$ .  
 (2)  $n \vee (n + m) = (n + 0) \vee (n + m) = n + (0 \vee m) = n + m$  and therefore  $n + m \geq n$ . Similarly  $n + m \geq m$  and consequently,  $n + m \geq n \vee m$ .  
 (3) If  $n + m \neq n \vee m$  then  $n + m > n \vee m$ . On contrary suppose that  $n \leq m$  then  $n \vee m = m$ . Therefore we have  $m < n + m$ . But then  $m + \vee\{m, n\} = m + m = m \neq n + m = \vee\{n + m, n + n\}$  implies that  $M$  is not an le-module. Hence  $n \not\leq m$  and similarly  $m \not\leq n$ .  
 (4) By the definition of le-module, we have  $(k + n) \vee (k + m) = k + (n \vee m) = k + m$ .

(5) We have  $n \wedge m \leq n$  then by (4),  $n \wedge m + n \wedge m \leq n \wedge m + n \leq n + n = n$ . Similarly  $n \wedge m + n \wedge m \leq m$ . Consequently,  $n \wedge m + n \wedge m \leq n \wedge m$ . If  $r \in R$  then  $r(n \wedge m) \leq rn \leq n$  and also  $r(n \wedge m) \leq m$  implies that  $r(n \wedge m) \leq n \wedge m$ . Hence  $n \wedge m$  is submodule element.

□

In particular, if 0 is the smallest element in  $M$ , i.e.,  $0 \leq n$  for all  $n \in M$  then  $n + m \geq n \vee m$  for all  $n, m \in M$

We call  $p \in Sub(M)$  as *prime element* if for  $r \in R$  and  $n \in M, rn \leq p$  implies  $re \leq p$  or  $n \leq p$ . We call  $q \in Sub(M)$  as *maximal element* if  $q \neq e$  and for any  $n \in Sub(M)$  with  $q \leq n \leq e$  either  $q = n$  or  $n = e$ . We denote the set of all maximal submodule elements by  $Max(M)$ .

1.4 Lemma:

[(Bhuniya and Kumbhakar, 2018, 2019)] Let  $M$  be an le-module then every maximal element in  $M$  is prime.

1.5 Lemma:

Let  $n, m \in Sub(M)$ . If  $n + m = e$  for all  $m \not\leq n$  then  $n \in Max(M)$ .

PROOF. If  $n \notin Max(M)$  then there exists  $k \in Sub(M)$  such that  $k > n$ . Therefore  $n + k = e$  but as  $k > n$  by Proposition [1.3(3)] we have  $n + k = k$ , a contradiction. □

In this paper we associate a graph  $G(M)$  on  $Sub(M)$  as follows: The vertex set of the graph  $G(M)$  is the set of all nonzero proper submodule elements of  $M$  and two distinct vertices  $n$  and  $m$  are adjacent if  $n + m = e$  and we call this graph as *associated graph of le-module*.

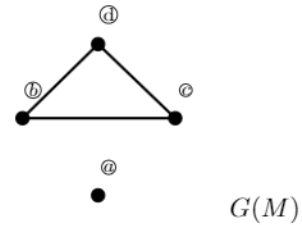
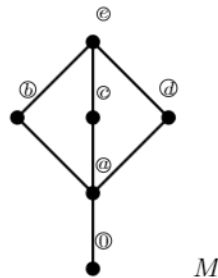
1.6 Remark:

Note that for  $n, m \in G(M)$ , if  $n + m = e$  and  $n \leq k$  then by Proposition [1.3(4)]  $k + m = e$  hence  $deg(n) \leq deg(k)$ .

1.7 Example:

Let  $M = \{0, a, b, c, d, e\}$  be an le-module over ring  $\mathbb{Z}_2$  with its lattice diagram and its graph with + is given in the table and usual multiplication with elements of the ring i.e  $0x = 0$  and  $1x = x$  for all  $x \in M$ . Note that  $G(M) = \{a, b, c, d\}$  and  $Sub(M) = M$ .

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	b	c	d	e
b	b	b	b	e	e	e
c	c	c	e	c	e	e
d	d	d	e	e	d	e
e	e	e	e	e	e	e



1.8 Definition:

A submodule element  $n$  of an le-module  $M$  is called *superfluous (or small)* if for every  $m \in Sub(M)$ ,  $n + m = e$  implies  $m = e$ . Note that 0 is always superfluous element but  $e$  is not superfluous.

1.9 Definition:

An le-module  $M$  is said to be a multiplication le-module if every submodule element  $n$  of  $M$  can be expressed as  $n = Ie$  for some ideal  $I$  of  $R$ , where  $Ie = \vee \{ \sum_{i=0}^k r_i e | k \in \mathbb{N}, r_i \in I \}$  [(Bhuniya and Kumbhakar, 2018, 2019)]

1.10 Lemma:

$M$  is multiplication le-module if and only if  $n = (n : e)e, \forall n \in Sub(M)$  where  $(n : e) = \{r \in R | re \leq n\}$ . [(Bhuniya and Kumbhakar, 2018, 2019)]

1.11 Definition:

An le-module  $M$  is called *coatomic* if for every  $n$  of  $M$  there exists  $m \in Max(M)$  such that  $n \leq m$ .

1.12 Definition:

An le-module  $M$  is called *simple* if  $0$  and  $e$  are the only submodule elements of  $M$ .

1.13 Definition:

The radical of an le-module  $M$  is the smallest submodule element if exists say,  $n \in M$ , such that  $n \geq m$  for every superfluous element  $m$  of  $M$ . Otherwise it is equal to  $e$ . We denote the radical of an le-module  $M$  by  $Rad(M)$ .

2. MAIN RESULTS:

2.1 Lemma:

Let  $M$  be a coatomic le-module then  $Rad(M)$  is a superfluous element and  $Rad(M) = \bigwedge_{m \in Max(M)} m$ .

PROOF. Let  $l$  be a superfluous element and there exists  $m \in Max(M)$  with  $l \not\leq m$ . Then  $l + m = e$  and  $l$  being superfluous implies  $m = e$ , a contradiction. Therefore  $l \leq m$  for all  $m \in Max(M)$  and hence  $l \leq \bigwedge_{m \in Max(M)} m$  for any superfluous element  $l$ . Thus we have  $Rad(M) \leq \bigwedge_{m \in Max(M)} m$ . It is enough to prove that  $\bigwedge_{m \in Max(M)} m$  is a superfluous element. Suppose  $\bigwedge_{m \in Max(M)} m + n = e$  for some  $n \in Sub(M)$  then  $n \leq m_j$  for some  $m_j \in Max(M)$  then  $\bigwedge_{m \in Max(M)} m + m_j = e$  but  $\bigwedge_{m \in Max(M)} m \leq m_j$  implies  $\bigwedge_{m \in Max(M)} m + m_j = m_j$ , a contradiction. Hence  $\bigwedge_{m \in Max(M)} m$  is a superfluous element and hence  $Rad(M) = \bigwedge_{m \in Max(M)} m$  □

2.2 Theorem:

Let  $M$  be a coatomic le-module and  $n, m \in Sub(M)$ . Then  $n \leq Rad(M)$  if and only if  $n$  is superfluous.

PROOF. Suppose that  $n \leq Rad(M)$  and  $n$  is not superfluous i.e., there exists  $m \in Sub(M)$  with  $n + m = e$  and  $m < e$ . Since  $M$  is coatomic, there exists  $k \in Max(M)$  with  $m \leq k$ . This implies that  $n + m \leq k + k = k < e$ , a contradiction. Consequently  $m = e$ .

Conversely suppose that  $n$  is superfluous and  $n \not\leq Rad(M)$ . Therefore there exists  $m \in Max(M)$  with  $n \not\leq m$ . Which implies  $n + m = e$  and since  $n$  is superfluous we have  $m = e$ , a contradiction. Consequently  $n \leq Rad(M)$ . □

2.3 Proposition:

[(Bhuniya and Kumbhakar, 2018, 2019)] Let  $M$  be an le-module and  $x \in M$ . Then for submodule elements  $k, l, n$  of  $M$ ,

- (1)  $l \leq n$  implies  $(l : x) \subseteq (n : x)$  and  $(k : n) \subseteq (k : l)$ ;
- (2)  $(l \wedge n : k) = (l : k) \cap (n : k)$

2.4 Theorem:

Let  $M$  be a multiplication le-module and  $n, m \in Sub(M)$ . If  $p$  is a prime submodule element of  $M$  with  $n \wedge m \leq p$ , then  $n \leq p$  or  $m \leq p$ .

PROOF. Suppose that  $p$  is a prime submodule element of  $M$  with  $n \wedge m \leq p$ . Then by Proposition[2.3]  $(n \wedge m : e) \subseteq (p : e)$  therefore  $(n \wedge m : e) = (n : e) \cap (m : e) \subseteq (p : e)$ . Let  $r_1 \in (n : e)$  and  $r_2 \in (m : e)$  with  $r_1, r_2 \notin (p : e)$ . Then  $r_1 e \leq n$  and  $r_2 e \leq m$  and this implies

$r_1 r_2 e \leq n \wedge m \leq p$ . But since  $p$  is prime, we have  $r_1 e \leq p$  or  $r_2 e \leq p$ , i.e.,  $r_1 \in (p : e)$  or  $r_2 \in (p : e)$ , a contradiction. Therefore  $(n : e) \subseteq (p : e)$  or  $(m : e) \subseteq (p : e)$ . Now, since  $M$  is a multiplication le-module, by Lemma[1.11] we have  $n = (n : e)e \leq (p : e)e = p$  or  $m = (m : e)e \leq (p : e)e = p$ . Consequently  $n \leq p$  or  $m \leq p$ .  $\square$

2.5 Corollary:

Let  $M$  be a coatomic multiplication le-module with  $Max(M) = \{m_i | i \in I\}$  and 0 as smallest element. Then for any nonempty proper finite subset  $\wedge$  of  $I$ , there exists proper submodule element, say  $m$  such that  $\wedge_{i \in I} m_i + m = e$

PROOF. On contrary suppose that there does not exist such  $m$ . Then by Theorem[2.2]  $\wedge_{i \in \wedge} m_i \leq Rad(M) \leq m_j$  for all  $j \in I$  and  $j \notin \wedge$ . Since every maximal element is a prime element by Lemma[1.5], then by Theorem[2.4] we have  $m_i \leq m_j$  for some  $i \in \wedge$ , a contradiction.  $\square$

Note that if  $\{M_i | 1 \leq i \leq n\}$  is a family of le-modules over ring  $R$  then  $M = \prod_{i=1}^n M_i$  is also an le-module over  $R$  with coordinate-wise addition, scalar multiplication and ordering. Also if  $m_i \in Sub(M_i)$  for  $i = 1, 2, \dots, n$  then  $\prod_{i=1}^n m_i \in Sub(M)$ . In the following theorem we have discussed the graph structure on product of simple le-modules

2.6 Theorem:

Let  $M = \prod_{i=1}^n M_i$  where each  $M_i$  is a simple le-module then  $G(M)$  is a connected  $n$ -partite graph.

PROOF. Let 0 and  $e_i$  be the only submodule elements of  $M_i$  for  $i = 1, 2, \dots, n$ . Then note that  $Sub(M) = \{\prod_{i=1}^n a_i | a_i = 0 \text{ or } e_i\}$ . Let  $\prod_{i=1}^n a_i$  and  $\prod_{i=1}^n b_i$  be any two nonzero proper submodule elements of  $M$ . If  $\prod_{i=1}^n a_i + \prod_{i=1}^n b_i = \prod_{i=1}^n e_i$  then  $\prod_{i=1}^n a_i$  and  $\prod_{i=1}^n b_i$  are adjacent. Suppose that they are not adjacent. Then we have following two cases:

Case i) If there exists  $j$  such that  $a_j = b_j = e_j$  then  $\prod_{i=1}^n c_i$  is a proper submodule element of  $M$  with  $c_i = e_i$  for  $i \neq j$  and  $c_j = 0$  for  $i = j$ . Then  $\prod_{i=1}^n a_i$  and  $\prod_{i=1}^n b_i$  are both adjacent to  $\prod_{i=1}^n c_i$

Case ii) If there does not exist  $j$  such that  $a_j = b_j = e_j$  then take  $\prod_{i=1}^n c_i$  such that  $c_i = 0$  if  $a_i = e_i$  and  $c_i = e_i$  if  $a_i = 0$ . Similarly, choose submodule element  $\prod_{i=1}^n d_i$  related to  $\prod_{i=1}^n b_i$ . Then  $\prod_{i=1}^n a_i$  is adjacent to  $\prod_{i=1}^n c_i$  and  $\prod_{i=1}^n b_i$  is adjacent to  $\prod_{i=1}^n d_i$ . Note that  $\prod_{i=1}^n c_i$  and  $\prod_{i=1}^n d_i$  are adjacent because  $c_i = d_i = 0$  and this will imply  $a_i = b_i = e_i$ .

Therefore  $G(M)$  is a connected graph. Now let  $V_j = \prod_{i=1}^n m_i$  with  $m_i = e_i$  for  $0 \leq i \leq j - 1$ ,  $m_j = 0$  and  $m_i = 0$  or  $e_i$  for  $j + 1 \leq i \leq n$ . Hence no two vertices of  $V_i$  are adjacent. Consequently  $G(M)$  is a connected  $n$ -partite graph.  $\square$

Sachin Ballal and Vilas Kharat studied Zariski topology on lattice modules [(Ballal and Kharat, 2015)]. In [(Ballal and Kharat, 2019)], they have topologize minimal spectrum of multiplication lattice modules. In [(Bhuniya and Kumbhakar, 2018)], Bhuniya and Kumbhakar studied prime spectrum of an le-module. Let  $Spec(M) = \{p \in M | p \text{ is a prime submodule element of } M\}$ . For  $n \in Sub(M)$ , we denote  $V(n) = \{p \in Spec(M) : n \leq p\}$  and  $\nu(M) = \{V(n) | n \in Sub(M)\}$ . If  $\nu(M)$  is closed under finite unions then there exists a topology on  $Spec(M)$  and we call this topology the quasi-Zariski topology and in this case  $M$  is called the top le-module. Note that associated graph on  $Spec(M)$  is a subgraph of  $G(M)$  and we denote it by  $G^{Spec}(M)$ .

In the following result, we establish a relationship between topology on  $Spec(M)$  and the graph on it.

2.7 Theorem:

Let  $M$  be a non-primeless top le-module. Then  $G^{Spec}(M)$  is a complete graph if and only if  $Spec(M)$  is a  $T_1$ -space.

PROOF. Suppose  $G^{Spec}(M)$  is a complete graph. Let  $p, q \in Spec(M)$  with  $q \in V(p)$ . This implies  $p \leq q$ . If  $p \neq q$  and  $G^{Spec}(M)$  is a complete graph then  $p + q = e$  implies  $q = e$ . For  $p \leq q$  implies  $q \leq p + q \leq q + q = q$ , i.e.  $p + q = q$ . But  $q \neq e$  implies  $V(p) = \{p\}$  is a closed set. Consequently  $Spec(M)$  is a  $T_1$ -space.

Conversely, suppose that  $Spec(M)$  is a  $T_1$ -space. Therefore,  $\{p\}$  is closed for all  $p \in Spec(M)$ . Note that  $\{p\} = \cap_j V(q_j)$ , where  $j \in I$  for some index set  $I$  and if  $p \leq q$  then  $q_j \leq p \leq q$  for all  $j$ . This implies  $q \in \cap_j V(q_j)$  and therefore  $p = q$  and hence  $V(p) = \{p\}$ . Thus every prime submodule element is maximal and hence  $p \neq q$  implies  $p + q = e$ . Consequently  $G^{Spec}(M)$  is a complete graph. □

Next Corollary establish the relation between  $Spec(R)$  and  $G^{Spec}(M)$ .

2.8 Corollary:

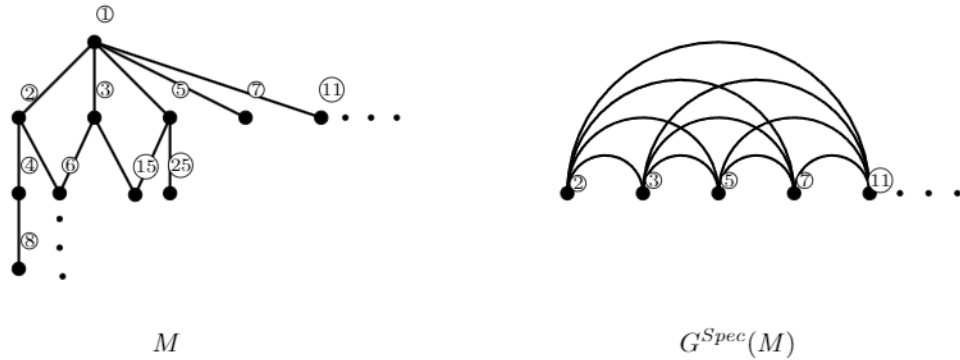
Let  $M$  be non-primeless multiplication le-module. If  $Spec(R)$  is a  $T_1$ -space then  $G^{Spec}(M)$  is a complete graph.

PROOF. Suppose that  $p, q \in Spec(M)$  with  $p \in \overline{\{q\}}$ . Then  $V(q) \supseteq \overline{\{q\}}$  which implies  $q \leq p$  and therefore  $(q : e) \subseteq (p : e)$ . Since  $Spec(R)$  is a  $T_1$ -space we have  $(q : e) = (p : e)$  and therefore by Lemma[1.11]  $q = (q : e)e = (p : e)e = p$  since  $M$  is a multiplication le-module. Hence  $\overline{\{q\}} = \{q\}$  i.e.,  $\{q\}$  is a closed set in  $Spec(M)$ . Therefore  $Spec(M)$  is a  $T_1$ -space. By Theorem[2.7]  $G^{Spec}(M)$  is a complete graph. □

The Converse of the Corollary[2.8] is not necessarily true. See the following example.

2.9 Example:

Let  $M = \mathbb{Z}$  over  $R = \mathbb{Z}$  be a multiplication le-module with respect to  $n + m = g.c.d(n, m)$ ,  $rn$  is the usual multiplication for  $r \in R, n \in M$  and  $m \geq n$  if and only if  $m$  divides  $n$ . Here note that  $Spec(R) = \{p\mathbb{Z} | p \text{ is a prime number}\}$  and  $Spec(M) = \{p \in \mathbb{Z} | p \text{ is a prime number}\}$ . Note that  $G^{Spec}(M)$  is a complete graph even though  $Spec(R)$  is not  $T_1$ -space.



2.10 Theorem:

Let  $M$  be an le-module. Then  $Rad(M) = 0$  if and only if  $G(M)$  is connected.

PROOF. Suppose that  $Rad(M) = 0$  and  $m_1, m_2 \in G(M)$  with  $m_1 \neq m_2$ . Note that  $m_1$  and  $m_2$  are not superfluous. Then there exist nonzero proper submodule elements  $n_1$  and  $n_2$  with  $m_1 + n_1 = e$  and  $m_2 + n_2 = e$ . Then  $m_1$  and  $n_1$  are adjacent and also  $m_2$  and  $n_2$  are adjacent. If  $n_1 + n_2 = e$  then  $n_1$  and  $n_2$  are adjacent and there is a path between  $m_1$  and  $m_2$ . If  $n_1 + n_2 \neq e$  then  $m_1 + (n_1 + n_2) = e$  and  $m_2 + (n_1 + n_2) = e$  and therefore there is a path between  $m_1$  and  $m_2$ . Consequently,  $G(M)$  is connected.

Conversely, suppose  $G(M)$  is connected and  $Rad(M) \neq 0$ . Then there exists nonzero superfluous element  $m$  such that  $m \leq Rad(M)$ . Since  $m$  is superfluous  $m + l = e$  for some  $l \in Sub(M)$  implies  $l = e$  i.e.,  $m$  is an isolated vertex of  $G(M)$ , a contradiction. Therefore  $Rad(M) = 0$   $\square$

2.11 Corollary:

Let  $M$  be a coatomic le-module. If  $G(M)$  is a tree then  $|Max(M)| = 2$ .

PROOF. Since  $G(M)$  is a tree,  $G(M)$  has no cycle which implies  $|Max(M)| < 3$  and by Theorem[2.10]  $G(M)$  is connected implies  $Rad(M) = 0$  and therefore  $|Max(M)| > 1$ , consequently  $|Max(M)| = 2$   $\square$

Following Theorem establish a Beck's conjecture for coatomic le-module.

2.12 Theorem:

Let  $M$  be a coatomic le-module. Then the clique number and the chromatic number of  $G(M)$  are equal to  $|Max(M)|$ .

PROOF. Let  $S$  be a complete subgraph of  $G(M)$ . For each vertex  $n$  of  $S$  there exists a maximal element  $m_n$  with  $n \leq m_n$ . For distinct vertices  $n_1$  and  $n_2$  of  $S$ , since  $n_1 + n_2 = e$ , we have  $m_{n_1} + m_{n_2} = e$  and which implies  $m_{n_1} \neq m_{n_2}$ . Thus the subgraph induced by  $T = \{m_n | n \text{ is a vertex of } S\}$  is a complete graph and  $|S| \leq |T|$ . Now  $G^{Max}(M)$  is a complete subgraph and  $|S| \leq |G^{Max}(M)|$  for complete subgraph  $S$ . Hence the clique number of  $G(M) = |Max(M)| = |G^{Max}(M)|$

Now to find the chromatic number of  $G(M)$ , let  $\{m_\lambda | \lambda \in \wedge\}$  be the set of all maximal submodule elements of  $M$ . For any  $\lambda \in \wedge$ , let  $G_\lambda(M) = \{n \in Sub(M) | 0 \neq n \leq m_\lambda, n \notin \bigcup_{\lambda' < \lambda} G_{\lambda'}(M)\}$ . Then for  $\lambda \in \wedge$ ,  $m_\lambda \in G_\lambda(M)$  and  $G_\lambda(M) \neq \emptyset$ . Also  $\{G_\lambda(M) | \lambda \in \wedge\}$  forms a partition for the set of all vertices of  $G(M)$ . Since for every  $\lambda \in \wedge$ , any two vertices in  $G_\lambda(M)$  are not adjacent, all vertices in  $G_\lambda(M)$  can have the same colour. However the  $m'_\lambda$ s must have different colours. Consequently the chromatic number of  $G(M) = |\wedge|$ .  $\square$

2.13 Definition:

Two vertices  $n$  and  $m$  are said to be orthogonal in  $G(M)$  if  $n + m = e$  and for every  $k \in G(M)$  either  $n + k \neq e$  or  $m + k \neq e$ .

2.14 Theorem:

If  $M$  is a coatomic le-module then the following statements are equivalent;

- (1)  $G(M)$  has no triangle.
- (2) Every two adjacent submodule elements are orthogonal.
- (3)  $M$  has at most two maximal submodule elements.

PROOF. (1)  $\implies$  (2) Suppose  $n, m \in G(M)$  with  $n + m = e$ . If  $m + k = e, n + k = e$  for some  $k \in G(M)$  then  $n - m - k - n$  forms a triangle in  $G(M)$  and therefore either  $n + k \neq e$  or  $m + k \neq e$ .

(2)  $\implies$  (3) If  $|Max(M)| \geq 3$ . Let  $m_1, m_2, m_3 \in Max(M)$  be distinct elements then it forms triangle of maximal submodule elements  $m_1 - m_2 - m_3 - m_1$  and we get non-orthogonal adjacent vertices  $\{m_1, m_2\}$ .

(3)  $\implies$  (1) If  $|Max(M)| = 1$  then we get empty graph. If  $|Max(M)| = 2 = |G(M)|$  then our graph is  $K_2$ . Now suppose  $|G(M)| \geq 3$  and  $Max(M) = \{m_1, m_2\}$  then for  $n \in Sub(M)$  we have  $n \leq m_1$  or  $n \leq m_2$ . Thus any two submodule elements of  $M$  are  $\leq m_1$  or  $\leq m_2$ . Without loss of generality if  $n \leq m_1$  and  $k \leq m_1$  then  $n + k \leq m_1 \neq e$ . Therefore  $G(M)$  has no triangle.  $\square$

2.15 Corollary:

In a coatomic le-module the girth of  $G(M)$  is always 3 except when  $|Max(M)| \leq 2$ .

2.16 Definition:

Let  $M$  be an le-module such that  $Max(M) \neq \emptyset$  and let  $S = \{n \in G(M) | n \not\leq Rad(M)\}$ . The subgraph generated by the set  $S$  is denoted by  $G^*(M)$ .

Note that, if  $Rad(M) = 0$  then  $G^*(M) = G(M)$

2.17 Theorem:

Let  $M$  be an le-module with  $Max(M) \neq \emptyset$ . Then  $G^*(M)$  is connected and  $diam(G^*(M)) \leq 3$ .

PROOF. Let  $m_1$  and  $m_2$  be two distinct elements of  $G^*(M)$ . Then there exist  $n_1, n_2 \in Max(M)$  such that  $m_1 \not\leq n_1, m_2 \not\leq n_2$ . If  $n_1 = n_2$  then  $m_1 + n_1 = e = m_2 + n_1$  and we have a path between  $m_1$  and  $m_2$ . If  $n_1 \neq n_2$  then  $n_1 + n_2 = e$  and we again have a path between  $m_1$  and  $m_2$ .  $\square$

2.18 Theorem:

Let  $M$  be an le-module with  $|Max(M)| = \infty$ . Then there exists  $n \in G^*(M)$  such that  $|Max(M) \setminus \nu(n)| = \infty$ , where  $\nu(n) = \{m \in Max(M) | n \leq m\}$ .

PROOF. If possible, suppose that for every  $n \in G^*(M)$ ,  $|Max(M) \setminus \nu(n)| < \infty$ . Let  $m_1$  and  $m_2$  be two distinct elements of  $G^*(M)$ . Then  $|Max(M) \setminus \nu(m_1)| < \infty$  and  $|Max(M) \setminus \nu(m_2)| < \infty$  and implies  $|\nu(m_1) \cap \nu(m_2)| = \infty$ . Therefore there exists  $q \in Sub(M)$  with  $m_1 \leq q$  and  $m_2 \leq q$ . But then  $m_1 + m_2 \leq q + q = q \neq e$  and which implies  $G^*(M)$  is totally disconnected, a contradiction to Theorem[2.17].  $\square$

2.19 Theorem:

Let  $M$  be a coatomic le-module. Then the following statements are equivalent.

- (1)  $G^*(M)$  is a complete bipartite graph.
- (2)  $|Max(M)| = 2$ .

PROOF. Suppose that  $G^*(M)$  is a complete bipartite graph with two parts say  $V_1$  and  $V_2$ . If  $|Max(M)| = 1$  then  $G^*(M)$  cannot be bipartite. Therefore  $|Max(M)| \geq 2$ . Suppose that  $|Max(M)| > 2$  then by the pigeon-hole principle two of the maximal elements belong to one of  $V_i$ , a contradiction to  $G^*(M)$  is a complete bipartite graph.

Conversely, suppose that  $Max(M) = \{n_1, n_2\}$ . Since  $M$  is coatomic, every submodule element of  $M$  is  $\leq n_1$  or  $\leq n_2$ . Let  $V_1 = \{n \in G^*(M) | n \leq n_1\}$  and  $V_2 = \{n \in G^*(M) | n \leq n_2\}$ . If  $n \in V_1 \cap V_2$  then  $n \leq n_1 \wedge n_2 = Rad(M)$ . But since  $n \in G^*(M)$ ,  $n \not\leq Rad(M)$ . Therefore  $V_1 \cap V_2 = \emptyset$  and  $G^*(M) = V_1 \cup V_2$ . Let  $m_1 \in V_1$  and  $m_2 \in V_2$  then  $m_1 + m_2 \not\leq n_1$ , otherwise  $m_2 \leq n_1$ , which is not true. Similarly  $m_1 + m_2 \not\leq n_2$ . Consequently,  $m_1 + m_2 = e$  and  $G^*(M)$  is a complete bipartite graph.  $\square$

2.20 Corollary:

Let  $M$  be a coatomic le-module with  $|Max(M)| > 1$  then  $G^*(M)$  is a star graph or girth of  $G(M) \leq 4$ .

PROOF. Let  $M$  be a coatomic le-module with  $|Max(M)| = 2$  then by Theorem[2.20],  $G^*(M)$  is a complete bipartite graph. Let  $V_1$  and  $V_2$  be two parts of the graph  $G^*(M)$ . If  $V_1$  or  $V_2$  contains single element then  $G^*(M)$  is a star graph. Otherwise we have a cycle  $v_{11} - v_{21} - v_{12} - v_{22} - v_{11}$  for  $v_{11}, v_{12} \in V_1$  and  $v_{21}, v_{22} \in V_2$  and hence the girth of  $G(M)$  is 4. If  $|Max(M)| > 2$  then by Corollary[2.15], the girth of  $G(M)$  is 3.  $\square$

2.21 Theorem:

If  $M$  is a coatomic le-module and  $|Max(M)| = n < \infty, n > 1$  then  $G^*(M)$  is  $n$ -partite

PROOF. Suppose that  $Max(M) = m_1, m_2, \dots, m_n$ . Since  $M$  is coatomic, every submodule element is  $\leq m_i$  for some  $i \in \{1, 2, \dots, n\}$ . Let  $W_i = \{n \in G^*(M) | n \leq m_i\}$  then take  $V_i =$



$W_i \setminus \cup_{j=1}^{i-1} W_j$ . If  $m_{i1}, m_{i2} \in V_i$  and  $m_{i1} + m_{i2} = e$  then  $e = m_{i1} + m_{i2} \leq m_i + m_i = m_i$ , a contradiction. Also  $m_i \in V_i$  implies  $V_i \neq \emptyset$  and  $M = V_1 \cup V_2 \cup \dots \cup V_n$ . Consequently,  $G^*(M)$  is  $n$ -partite  $\square$

2.22 Theorem:

Let  $M$  be an le-module and  $G^*(M)$  be a star graph. Then  $|Max(M)| = 2$  and  $M$  is coatomic.

PROOF. Since  $G^*(M)$  is a star graph implies  $|Max(M)| < 3$  and there exists a vertex  $n \in G^*(M)$  such that  $n$  is adjacent to all other vertices. Then  $n \in Max(M)$ . For, if  $k \in G(M)$  with  $n \leq k$  then  $n + k \leq k + k = k \neq e$  and it implies that  $n$  and  $k$  are not adjacent and therefore  $k \notin G^*(M)$ , a contradiction to the fact that  $n \in G^*(M)$ . Note that  $|Max(M)| \neq 1$ , otherwise  $Rad(M) = n$  and  $n \notin G^*(M)$ . Now suppose that  $m \in Max(M)$ . If  $s \not\leq n$  and  $s \not\leq m$  for some  $s \in Sub(M)$  then  $s + n = e, s + m = e$  then  $s \in G^*(M)$  and  $s - n - m - s$  is a cycle in  $G^*(M)$ , a contradiction. Hence  $M$  must be coatomic and  $|Max(M)| = 2$ .  $\square$

2.23 Theorem:

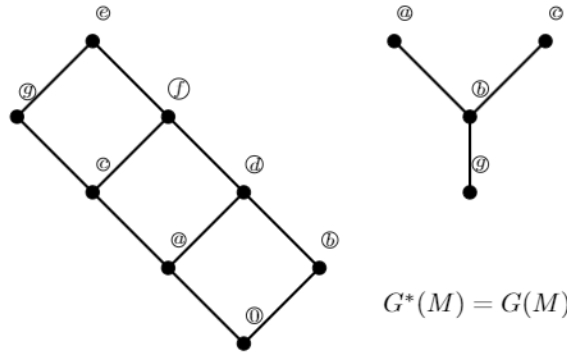
If  $M$  is coatomic le-module then  $G^*(M)$  is a star graph if and only if  $G^*(M)$  is a tree graph.

PROOF. If  $G^*(M)$  is a star graph then clearly it is a tree graph. Conversely suppose that  $G^*(M)$  is a tree graph then  $|Max(M)| < 3$  because tree contains no cycle. If  $|Max(M)| = 1$  then the graph of  $G^*(M)$  is empty. If  $|Max(M)| = 2$  then by Theorem[2.19],  $G^*(M)$  is a complete bipartite graph. Therefore  $G^*(M)$  is complete bipartite and tree. Hence  $G^*(M)$  is a star graph  $\square$

2.24 Example:

Let  $M = \{0, a, b, c, d, g, f, e\}$  be an le-module over  $\mathbb{Z}_2$  with '+' as given in the table and usual multiplication of ring  $\mathbb{Z}_2$  with elements of  $M$ , i.e.,  $0x = 0$  and  $1x = x$  for all  $x \in M$  and note that  $Sub(M) = \{0, a, b, c, g, e\}$ . Here  $G^*(M) = G(M)$  as  $Rad(M) = 0$  and  $G^*(M)$  is a star graph and hence complete bipartite graph as shown in the following Figure.

+	0	a	b	c	d	g	f	e
0	0	a	b	c	d	g	f	e
a	a	a	e	c	e	g	e	e
b	b	e	b	e	e	e	e	e
c	c	c	e	c	e	g	e	e
d	d	e	e	e	e	e	e	e
g	g	g	e	g	e	g	e	e
f	f	e	e	e	e	e	e	e
e	e	e	e	e	e	e	e	e



$M$

2.25 Theorem:

If  $M$  is a coatomic le-module and  $G^*(M)$  contains a cycle then the core of  $G$  is a union of triangles and rectangles, and every vertex of  $G^*(M)$  is either an end vertex or a vertex of the core.

PROOF. Suppose  $(m_1, m_2, \dots, m_n, m_1)$  is a cycle. If  $n \leq 4$  then the result holds trivially. If  $n \geq 5$  and  $m_1 + m_3 = e$  or  $m_2 + m_{n-1} = e$  or  $m_2 + m_n = e$  then  $m_1 - m_2$  belongs to a triangle or rectangle. Assume that  $m_1 + m_3 \neq e, m_2 + m_{n-1} \neq e$  and  $m_2 + m_n \neq e$ .

Case i) Suppose  $m_1 + m_{n-1} \neq e$ . Then there exists  $m \in \text{Max}(M)$  with  $m_1 + m_{n-1} \leq m$  and hence  $m_1 + m_{n-1} + m_2 \leq m + m_2$  and therefore  $m + m_2 = e$ . Similarly,  $m + m_n = e$  and consequently  $(m_1, m_2, m, m_n, m_1)$  is a rectangle.

Case ii) Suppose  $m_1 + m_{n-1} = e$ . Here we will use mathematical induction. If  $n = 5$  then  $(m_1, m_2, m_3, m_4, m_1)$  is a cycle. Assume the result for  $n = k$ . Then,  $(m_1, m_2, \dots, m_{k-1}, m_1)$  we have  $m_1 - m_2$  belongs to a triangle or a rectangle. If  $(m_1, m_2, \dots, m_{k-1}, m_k, m_1)$  is a cycle then by assumption  $m_1 + m_{k-1} = e$  and which implies that  $(m_1, m_2, \dots, m_{k-1}, m_1)$  is a cycle and therefore by the induction result follows.

Suppose that  $m$  is not a vertex of a cycle. We prove that only one edge is adjacent to  $m$ . If possible, suppose  $m_1, m_2$  are two vertices adjacent to  $m$  then there exists a path  $m_1 - m - m_2 - c$  since  $G^*(M)$  is connected. If  $m_3 = m_1 + c$  and  $m_3 = e$ , then  $(m_1, m, m_2, c, m_1)$  is a cycle and therefore  $m_3 \neq e$ . But then  $m_2 + m_3 = m_2 + m_1 + c = e + m_1 = e$  and this implies  $m + m_3 = m + m_1 + c = e + c = e$ , consequently  $(m, m_2, m_3, m)$  is a cycle, a contradiction. Therefore  $m$  is an end vertex.  $\square$

2.26 Theorem:

[(Bhuniya and Kumbhakar, 2019)] Let  $p$  be a prime submodule element of an le-module  $M$  and  $x \in M$ . Then  $(p : x)$  is a prime ideal of  $R$  for every  $x \in M$ .

2.27 Theorem:

Let  $M$  be a coatomic le-module over a ring  $R$ . If for  $n, m \in \text{Max}(M)$  with  $(m : e) \not\subseteq (n : e)$  then  $G^*(M)$  is complete if and only if  $G^*(M) = K_2$ .

PROOF. If  $|\text{Max}(M)| = 1$  then  $G^*(M)$  is an empty graph. Suppose  $|\text{Max}(M)| > 2$ . Let  $m_1, m_2, m_3 \in \text{Max}(M)$  be distinct elements. We prove that  $(m_1 : e)m_2 \in G^*(M)$ . If possible, suppose  $(m_1 : e)m_2 \leq \text{Rad}(M)$  then  $(m_1 : e)m_2 \leq m_3$ . Let  $r_1 \in (m_1 : e)$  and  $r_2 \in (m_2 : e)$  implies  $r_1e \leq m_1$  and  $r_2e \leq m_2$ . Therefore  $r_1r_2e \leq r_1m_2 \leq m_3$  implies  $r_1r_2 \in (m_3 : e)$  and hence  $(m_1 : e)(m_2 : e) \subseteq (m_3 : e)$ . But  $m_3$  is a maximal element and hence prime, by Theorem[2.26],  $(m_3 : e)$  is a prime ideal. Therefore  $(m_1 : e) \subseteq (m_3 : e)$  or  $(m_2 : e) \subseteq (m_3 : e)$ , a contradiction to the assumption. Hence  $(m_1 : e)m_2 \not\leq m_3$  and therefore we have  $(m_1 : e)m_2 \in G^*(M)$ . Note that  $(m_1 : e)m_2 \leq m_1$  implies  $(m_1 : e)m_2 + m_1 = m_1 \neq e$ . Hence  $d((m_1 : e)m_2, m_1) \neq 1$ . Now  $(m_1 : e)m_2 + m_3 = e$  and therefore  $d((m_1 : e)m_2, m_1) = 2$ , but  $\text{diam}(G^*(M)) = 1$  because  $G^*(M)$  is complete. Therefore  $|\text{Max}(M)| = 2$ . Then by Theorem[2.19],  $G^*(M)$  is a complete bipartite graph. Thus  $G^*(M) = K_2$ , because it is complete and complete bipartite graph.

$\square$

References

A. ABBASI, H. ROSHAN-SHEKALGOURABI, D. H.-L. 2015. Associated graphs of modules over commutative rings. *Iranian Journal of Mathematical Sciences and Informatics Vol.10(1)*, 45-58.

BALLAL, S. AND KHARAT, V. 2015. Zariski topology on lattice modules. *Asian-European Journal of Mathematics Vol.8(4)*.

BALLAL, S. AND KHARAT, V. 2019. On minimal spectrum of multiplication lattice modules. *Mathematica Bohemica Vol.144(1)*, 85-97.

BECK, I. 1988. Coloring of a commutative ring. *J. Algebra. Vol.116*, 208-226.

BHUNIYA, A. K. AND KUMBHAKAR, M. 2018. On irreducible pseudo-prime spectrum of topological le-modules. *Quasigroups and Related Systems Vol.26(2)*, 251-262.

BHUNIYA, A. K. AND KUMBHAKAR, M. 2019. Uniqueness of primary decompositions in laskerian le-modules. *Acta Math. Hunga. Vol.158(1)*, 202-215.

BONDY, J. A. AND MURTY, U. S. R. 2008. Graph theory. *Gradute Text in Mathematics, 244, Springer, New York*.

NARAYAN PHADATARE, S. B. AND KHARAT, V. 2019. Semi-complement graph of lattice modules.  
*Soft Computing. Vol.23*, 3973-3978.

**Sadashiv Puranik** is working as an Assistant Professor and Head Department of Mathematics at Tuljaram Chaturchand College of Arts, Science and Commerce, Baramati, Dist-Pune. He has completed his M.Sc. in Mathematics from Department of Mathematics, University of Pune. He is pursuing his Ph.D. at Department of Mathematics, Savitribai Phule Pune University, Pune (Formerly University of Pune). His research interest is in Algebra and Analysis. He is also a member of Indian Mathematical Society.



**Sachin Ballal** did his post graduation and Doctor of Philosophy in Mathematics from University of Pune. He joined University of Pune as an Assistant Professor in 2008. He is recipient of Gold Medal from University of Pune for best Ph.D. thesis in Mathematics for the year 2016-17, and also recipient of best research paper award by Vijnāna Parishad of India in 2018. He has published 11 research papers in international journals. One Ph.D. student is working under his guidance. He is a member of Indian Mathematical Society and also a member of Board of Studies of various Universities.



**Vilas Kharat** graduated with Physics, Chemistry and Mathematics, subsequently did his Master of Science from the Dr. B. A. Ambedkar Marathwada University, Aurangabad and Doctor of Philosophy from the University of Pune. During late Eighties he joined as a lecturer and in mid Nineties University of Pune, became full Professor in 2005. He is recipient of four consecutive best research papers awards by Indian Mathematical Society for the years 1998, 1999, 2000 and 2001. Professor Kharat has published number of research papers in different international journals. Five students have obtained their Ph.D. degrees and work of seven is in progress and importantly he has to his credit a partial solution to a very famous Frankl's Conjecture in collaboration with his research student. Kharat is also member of various learned societies including IEEE, American Mathematical Society member of various academic bodies which includes Board of studies, Research and Recognition Committee, Faculty of Science of various universities.

